

Geometry of the uniform spanning forest components in high dimensions

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Abstract

In this note we study the geometry of the component of the origin in the Uniform Spanning Forest of \mathbb{Z}^d , as well as in the Uniform Spanning Tree of wired subgraphs of \mathbb{Z}^d , when $d \geq 5$. In particular, we study connectivity properties with respect to the Euclidean and the intrinsic distance. We intend to supplement these with further estimates in the future. We are making this preliminary note available, as one of our estimates is used in work of Bhupatiraju, Hanson and Járai [BHJ] on sandpiles.

1 Introduction

The Uniform Spanning Tree (UST) on a finite graph G is a random spanning tree of G , chosen uniformly among all spanning trees of G . Motivated by questions of Lyons, Pemantle [Pem91] considered the weak limit of the USTs on a growing sequence of subgraphs of \mathbb{Z}^d , induced by sets $V_n \uparrow \mathbb{Z}^d$, and showed that the limit exists. The limiting random object, that is a random spanning forest of \mathbb{Z}^d , is called the Uniform Spanning Forest (USF). Implicit in Pemantle's work is the result that an alternative choice of boundary condition yields the same limit. Namely, form the “wired” graph $G_n^W = (V_n \cup \{r_n\}, E_n)$, by collapsing all vertices in $\mathbb{Z}^d \setminus V_n$ into r_n , and removing self-loops created at r_n . Then the weak limit of the USTs on G_n^W coincides with the USF. One of Pemantle's results was that the USF is connected a.s. in dimensions $1 \leq d \leq 4$, but it consists of infinitely many (infinite) trees a.s. in dimensions $d \geq 5$.

Fundamental to the study of the UST/USF is Wilson's algorithm [W], [LP] that allows one to build the UST/USF from Loop-Erased Random Walks (LERWs), and thereby analyze it in terms of random walk. All the necessary background about the UST/USF, that we do not detail in this note, can be found in the book [LP].

Masson [Mas] and Barlow and Masson [BM1, BM2] studied the geometry of the LERW and the UST in two dimensions. This led to a detailed understanding of random walk on the UST. The purpose of this note is to prove estimates on the geometry of the LERW

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and the USF in dimensions $d \geq 5$. We are interested in properties such as the length of paths and volumes of balls, both with respect to Euclidean distance and the intrinsic metric of the tree components. On the one hand we are interested in extending results from 2D to high dimensions, where the geometry is very different. On the other hand, our Theorem 5.4 is used in work of Bhupatiraju, Hanson and J  rai [BHJ] on sandpiles.

2 Notation

Let $\mathcal{U} = \mathcal{U}_{\mathbb{Z}^d}$ be the USF in \mathbb{Z}^d , viewed as a random subgraph of the nearest neighbour integer lattice. Write $\mathcal{U}(x)$ for the connected component of \mathcal{U} containing x .

We extend this notation to $D \subset \mathbb{Z}^d$ as follows. When D is finite, $\mathcal{U} = \mathcal{U}_D$ denotes the UST on the wired graph $G_D^W = (D \cup \{r_D\}, E_D)$, where E_D can be identified with those edges of \mathbb{Z}^d that have at least one endpoint in D . For $x \in D$, we denote by $\mathcal{U}(x)$ the connected component of x in the graph obtained from \mathcal{U} by splitting all edges at r_D . In other words, $\mathcal{U}(x)$ is the union of those paths in \mathcal{U} that do not contain r_D as an interior vertex. We write \mathcal{U}_0 for $\mathcal{U}(0)$, when $0 \in D$.

When $D \subset \mathbb{Z}^d$ is infinite, we let \mathcal{U} denote the weak limit, as $n \rightarrow \infty$, of the USTs on the wired graphs $G_{D_n}^W$, where $D_n = \{x \in D : |x| \leq n\}$. The limit exists due to monotonicity; see [LP]. Wilson’s algorithm rooted at infinity [BLPS], [LP] can be easily adapted to sample \mathcal{U} . We let $\mathcal{U}(x)$ denote the union of those paths in \mathcal{U} that do not contain r_D as an interior vertex, and $\mathcal{U}_0 = \mathcal{U}(0)$.

For any of the cases of \mathbb{Z}^d , or $D \subset \mathbb{Z}^d$ finite or infinite, we let

$$d_{\mathcal{U}}(x, y) := \text{graph distance between } x \text{ and } y \text{ in } \mathcal{U},$$

where, if $y \notin \mathcal{U}(x)$, we set $d_{\mathcal{U}}(x, y) = \infty$. The meaning of \mathcal{U} will always be clear from context.

Notation for sets: We denote balls in different metrics as follows:

$$\begin{aligned} B_E(x, r) &= \{y \in \mathbb{Z}^d : |x - y| \leq r\}, \\ B_n &= B_E(0, n) \\ Q(x, n) &= \{y \in \mathbb{Z}^d : \|x - y\|_{\infty} \leq n\}, \\ Q_n &= Q(0, n), \\ B_{\mathcal{U}}(x, r) &= \{y \in \mathbb{Z}^d : d_{\mathcal{U}}(x, y) \leq r\}, \end{aligned}$$

For $A \subset \mathbb{Z}^d$ we denote:

$$\begin{aligned} \partial A &= \{x \in \mathbb{Z}^d - A : x \sim y \text{ for some } y \in A\}, \\ \partial_i A &= \{x \in A : x \sim y \text{ for some } y \in A^c\}, \\ A^o &= A - \partial_i A. \end{aligned}$$

Let π_i be projection onto the i th coordinate axis, and \mathbb{H}_n be the hyperplane

$$\mathbb{H}_n = \{x : \pi_1(x) = n\}.$$

Let $\mathcal{R}_n = \{n\} \times [-n, n]^{d-1}$ denote the “right-hand face” of $[-n, n]^d$, in the first coordinate direction.

Notation for processes. $S^x = (S_k^x, k \geq 0)$ is simple random walk with $S_0^x = x$, and \mathbb{P}^x is its law. We let $S = S^0$, and $\mathbb{P} = \mathbb{P}^0$. If we discuss random walks S^x and S^y with $x \neq y$, then they will always be independent.

A path γ is a (non-necessarily self avoiding) sequence of adjacent vertices in \mathbb{Z}^d – ie $\gamma = (\gamma_0, \gamma_1, \dots)$ with $\gamma_{i-1} \sim \gamma_i$. (Sometimes we will write $\gamma(i)$ for γ_i .) Paths can be either finite or infinite. We will often need to consider the beginning or final portions of paths with respect to the first or last hit on a set. To this end, we define a number of operations on paths. Let $\gamma = (\gamma_0, \gamma_1, \dots)$ be a path. Given a set $A \subset \mathbb{Z}^d$ define $k_1 = \min\{k \geq 0 : \gamma_k \in A\}$, $k_2 = \max\{k \geq 0 : \gamma_k \in A\}$, and set

$$\begin{aligned}\mathcal{B}_A^F \gamma &= (\gamma_{k_1}, \gamma_{k_1+1}, \dots), \\ \mathcal{B}_A^L \gamma &= (\gamma_{k_2}, \gamma_{k_2+1}, \dots), \\ \mathcal{E}_A^F \gamma &= (\gamma_0, \dots, \gamma_{k_1}), \\ \mathcal{E}_A^L \gamma &= (\gamma_0, \dots, \gamma_{k_2}), \\ \Theta_k \gamma &= (\gamma_k, \dots), \\ \Phi_k \gamma &= (\gamma_0, \dots, \gamma_k), \\ H_A(\gamma) &= \sum_i 1_{(\gamma_i \in A)}.\end{aligned}$$

Thus $\mathcal{B}_A^F \gamma$ is the path γ ‘Beginning’ at the ‘First’ hit on A , and $\mathcal{E}_A^L \gamma$ is the path γ ‘Ended’ at the ‘Last’ hit on A , etc. If γ is a finite path we write $|\gamma|$ for the length of γ . $H_A(\gamma)$ is the number of hits by γ on the set A . Let $\mathcal{L}\gamma$ be chronological loop erasure of γ , and if $\gamma = (\gamma_0, \dots, \gamma_n)$ is a finite path let $\mathcal{R}\gamma = (\gamma_n, \gamma_{n-1}, \dots, \gamma_0)$ be the time reversal of γ .

We define hitting times

$$\begin{aligned}\tau_A &= \inf\{j \geq 0 : S_j \notin A\}, \\ T_A &= \inf\{j \geq 0 : S_j \in A\}, \\ T_A^+ &= \inf\{j \geq 1 : S_j \in A\}.\end{aligned}$$

When we need to specify the process we write $T_A[S]$ etc.

Given a domain $D \subset \mathbb{Z}^d$, we denote the Green functions

$$\begin{aligned}G_D(x, y) &= \mathbb{E}^x \left(\sum_{0 \leq k < \tau_D} I[S^x_k = y] \right) \\ G(x, y) &= G_{\mathbb{Z}^d}(x, y).\end{aligned}$$

A note on constants. Throughout, c and C will denote positive finite constants that only depend on the dimension d , and whose value may change from line to line, and even within a single string of inequalities.

3 Properties of the LERW

In this section we derive a number of auxiliary estimates on LERW in dimensions $d \geq 5$. Some of these will be used in Sections 4 and 5, where we give upper and lower bounds on

the volume of balls in the intrinsic metric. Two results of this section that are of interest in themselves are: (i) Proposition 3.11, that gives a large deviation upper bound on the lower tail of the number of steps in a LERW up to its exit from a large box; and (ii) Theorem 3.12, that gives an upper bound on the probability that $x, y \in \mathbb{Z}^d$ are in the same component of \mathcal{U} and the path between them has length at most n .

The papers [Mas, BM1] give a number of properties of LERW in \mathbb{Z}^2 , some of which hold for more general graphs.

A fundamental fact about LERWs is the following “Domain Markov property” — see [La2].

Lemma 3.1. *Let $D \subset \mathbb{Z}^d$, let $\gamma = (\gamma_0, \dots, \gamma_n)$ be a path from $x = \gamma_0$ to D^c . Set $\alpha = \Phi_k \gamma$, $\beta = \Theta_k \gamma$. Let Y be a random walk started at γ_k conditioned on the event $\{\tau_D(Y) < T_\alpha^+(Y)\}$. Then*

$$\mathbb{P}(\mathcal{L}(\mathcal{E}_{D^c}^F S) = \gamma | \Psi_k(\mathcal{L}(\mathcal{E}_{D^c}^F S)) = \alpha) = \mathbb{P}(\mathcal{L}(\mathcal{E}_{D^c}^F Y) = \beta). \quad (3.1)$$

A key result in [Mas] is a ‘separation lemma’ when $d = 2$ — see [Mas, Theorem 4.7]. Let S, S' be independent SRW in \mathbb{Z}^d with $S_0 = S'_0 = 0$, and T_n, T'_n be the hitting times of ∂Q_n . Set

$$\begin{aligned} F_n &= \{S[1, T_n] \cap S'[1, T'_n] = \emptyset\}, \\ Z_n &= d(S(T_n), S'[1, T'_n]) \vee d(S'(T'_n), S[0, T_n]). \end{aligned}$$

Lemma 3.2. (*‘Separation lemma’*). *Let $d \geq 5$. There exists $c_1 > 0$ such that*

$$\mathbb{P}(Z_n \geq \tfrac{1}{2}n | F_n) \geq c_1.$$

Proof. Let $e_1 = (1, 0, \dots, 0)$. Let X be a SRW started at $2ke_1$, and $A_k = \{je_1, k \leq j \leq 2k\}$. Since $d \geq 5$ two independent SRWs intersect with probability less than 1, and thus there exists k (depending on d) such that

$$\mathbb{P}^0(S \text{ hits } X \cup A_k) \leq \tfrac{1}{16}d^{-2}.$$

Now fix this k , and let

$$G_1 = \{S_i = -ie_1, S'_i = ie_1, 0 \leq i \leq k\}.$$

So $\mathbb{P}(G_1) = (2d)^{-2k}$. Then writing $G_2 = \{S[1, T_{n/2}] \cap S'[1, T'_{n/2}] \neq \emptyset\}$,

$$\begin{aligned} \mathbb{P}(G_2 | G_1) &\leq \mathbb{P}(S[k+1, T_{n/2}] \cap S'[1, T'_{n/2}] \neq \emptyset | G_1) + \mathbb{P}(S[1, T_{n/2}] \cap S'[k, T'_{n/2}] \neq \emptyset | G_1) \\ &\leq \tfrac{1}{8}d^{-2}. \end{aligned}$$

Let H_\pm be the left and right faces (in the e_1 direction) of the cube $Q_{n/2}$. We have

$$\mathbb{P}(S_{T_{n/2}} \in H_- | G_1) \geq (2d)^{-1}.$$

So if $G_3 = G_2^c \cap \{S_{T_{n/2}} \in H_-, S'_{T'_{n/2}} \in H_+\}$,

$$\begin{aligned} \mathbb{P}(G_3|G_1) &\geq \mathbb{P}(S_{T_{n/2}} \in H_-, S'_{T'_{n/2}} \in H_+|G_1) - \mathbb{P}(G_2|G_1) \\ &\geq (2d)^{-2} - (8d^2)^{-1} = (8d^2)^{-1}. \end{aligned}$$

If G_3 occurs then let G_4 be the event that S' then (i.e. after time $T'_{n/2}$ leaves Q_n before it hits \mathbb{H}_0 , and S leaves Q_n before it hits \mathbb{H}_0 . By comparison with a one-dimensional SRW each of these events has probability at least $1/3$, so $\mathbb{P}(G_4|G_3) \geq 1/9$. On the event $G_1 \cap G_3 \cap G_4$ the path $S[0, T_n]$ is contained in $[-n, 0] \times [-n, n]^{d-1} \cup Q_{n/2}$, and $\pi_1(S'_{T'_n} = n)$, so that $d(S'_{T'_n}, S[0, T_n]) \geq n/2$. The same bound holds if we interchange S' and S , and so we deduce that

$$\mathbb{P}(Z_n \geq \tfrac{1}{2}n | F_n) \geq \mathbb{P}(\{Z_n \geq \tfrac{1}{2}n\} \cap F_n) \geq \mathbb{P}(G_1 \cap G_3 \cap G_4) \geq (2d)^{-2k} (8d^2)^{-1} 9^{-1}.$$

□

Remark. The result in $d \geq 5$ is much easier than $d = 2$, since with high probability S and S' do not intersect. The proof for $d = 2$ uses the fact that if the two processes get too close, then by the Beurling estimate they hit with high probability.

In the remainder of this section we give some estimates on the length of LERW paths in \mathbb{Z}^d with $d \geq 5$. We fix $D \subset \mathbb{Z}^d$ and $N \geq 1$ such that $Q_N = Q(0, N) \subset D$. We will be interested in the number of steps the LERW from 0 to ∂D takes up to its first exit from Q_N . Let S be SRW on \mathbb{Z}^d with $S_0 = 0$. Let

$$L = \mathcal{L}(\mathcal{E}_{D^c}^F(S)).$$

In words, L is the loop erasure of S up to its first hit on the boundary of D .

Our estimate will be broken down into studying L in ‘shells’ $Q_{n+m} \setminus Q_n$. For this purpose, let us fix n, m such that $16 \leq n < n+m \leq N$, with $m \leq n/8$. Let

$$\alpha = \mathcal{E}_{\partial_i Q_n}^F L, \quad L' = \mathcal{B}_{\partial_i Q_n}^F L.$$

So α is the path L up to its first hit on $\partial_i Q(0, n)$, and L' is the path of L from this time on. See Figure 1.

Let us condition on α . Let $x_0 \in \partial_i Q_n$ be the endpoint of α . When $x_0 \in \mathbb{H}_n$, we let $x_1 = x_0 + (m/2)e_1$ and set

$$A = A(x_0) = Q(x_1, m/4), \quad A^* = Q(x_1, 3m/8).$$

When x_0 lies on one of the other faces of Q_n , we replace e_1 by the unit vector pointing towards that faces to define x_1 and $A(x_0)$. See Figure 1. Set

$$\beta = \mathcal{E}_{\partial_i Q(x_0, m)}^F L'.$$

Let \tilde{X}^z be S^z conditioned on $\{\tau_D < T_\alpha^+\}$. While the process \tilde{X}^z depends on α , our notation will not emphasize this point. Write \tilde{X} for \tilde{X}^{x_0} , and $\tilde{G}_D(x, y)$ for the Green

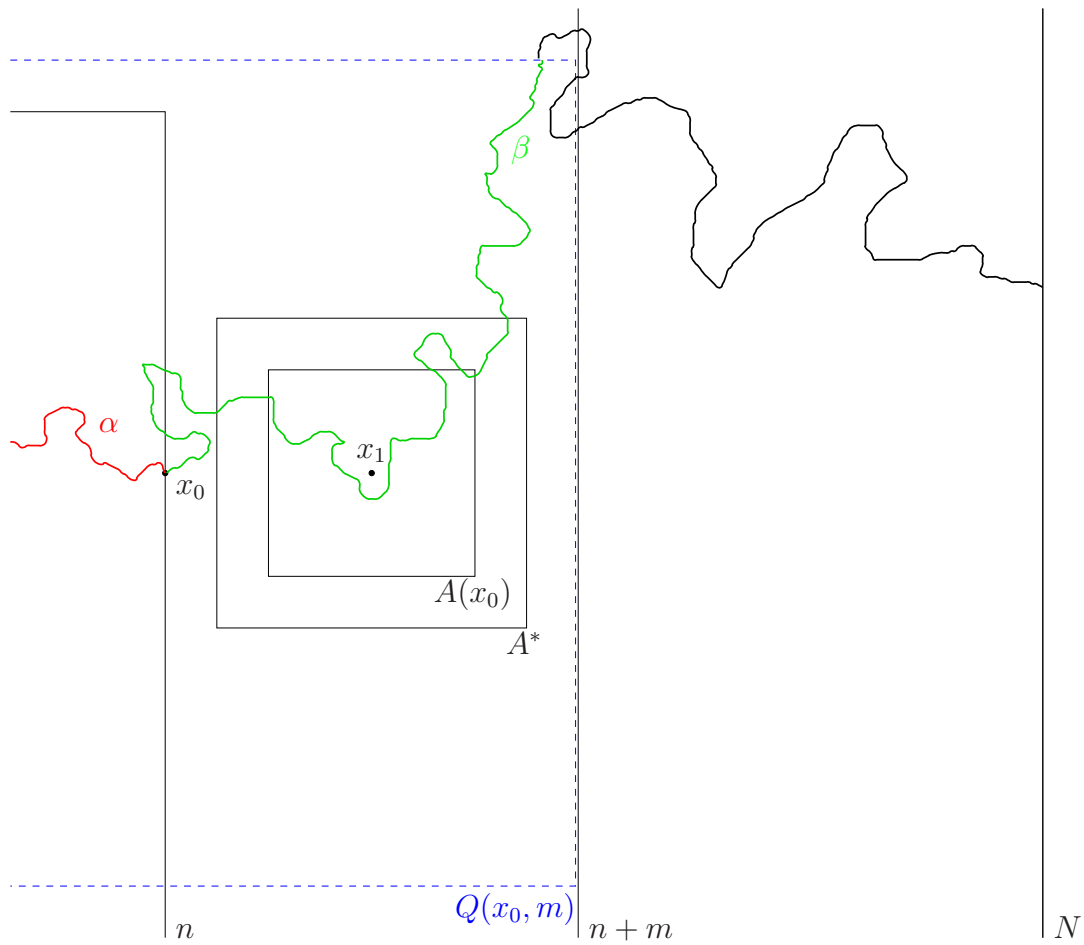


Figure 1: Setup and notation for the piece of the LERW in the shell $Q_{n+m} \setminus Q_n$.

function for \tilde{X}^x . By the domain Markov property, Lemma 3.1, we have (conditional on α) that

$$L' \stackrel{(d)}{=} \mathcal{L}(\mathcal{E}_{\partial D}^F \tilde{X}). \quad (3.2)$$

We write \tilde{T} , $\tilde{\tau}$, etc. for hitting and exit times by \tilde{X} . Set

$$h(x) = \mathbb{P}^x(\tau_D < T_\alpha).$$

Then

$$\tilde{G}_D(x, y) = \frac{h(y)}{h(x)} G_D(x, y), \quad x, y \in D - \alpha. \quad (3.3)$$

The standard Harnack inequality (see [La2]) gives

$$h(y) \asymp h(x_1), \quad y \in A^*, \quad (3.4)$$

and thus

$$\tilde{G}_D(x, y) \asymp G_D(x, y), \quad x, y \in A^*. \quad (3.5)$$

Lemma 3.3. *Let $d \geq 3$. For any α we have*

$$\mathbb{E}(H_A(\beta)|\alpha) \leq c_1 m^2, \quad (3.6)$$

$$\mathbb{E}(H_A(\beta)^2|\alpha) \leq c_1 m^4. \quad (3.7)$$

Proof. This is a standard computation with Green functions. Let $B = Q(x_0, m)$. Then, since β is a subset of the path of \tilde{X} , we have

$$H_A(\beta) \leq \sum_{k=0}^{\tilde{\tau}_B} 1_{(\tilde{X} \in A)} = H_A(\mathcal{E}_{\partial_i B}^F \tilde{X}) =: \tilde{H}.$$

Then for $p = 1, 2$,

$$\mathbb{E}^{x_0}(\tilde{H}^p|\alpha) = \mathbb{E}^{x_0}\left(1_{(\tilde{T}_{A^*} < \tilde{\tau}_B)} \mathbb{E}^{\tilde{X}_{\tilde{T}_{A^*}}}(\tilde{H}^p)\right) \leq \max_{z \in \partial_i A^*} \mathbb{E}^z \tilde{H}^p.$$

Let $z \in \partial_i A^*$. Then using (3.5)

$$\mathbb{E}^z(\tilde{H}|\alpha) = \sum_{y \in A} \tilde{G}_B(z, y) \leq c|A| \max_{y \in A} G_B(z, y) \leq c' m^2 m^{2-d} = c' m^2.$$

Also since on A^* we have $\tilde{G}_B \asymp G_B \leq G$,

$$\begin{aligned} \mathbb{E}^z(\tilde{H}^2|\alpha) &\leq 2 \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 1_{(k \leq \tilde{\tau}_B)} 1_{(j \leq \tilde{\tau}_B)} 1_{(\tilde{X}_k \in A)} 1_{(\tilde{X}_j \in A)} \\ &\leq 2 \sum_{x \in A} \sum_{y \in A} \tilde{G}_B(z, x) \tilde{G}_B(x, y) \\ &\leq c|A| m^{2-d} \max_{x \in A} \sum_{y \in A} \tilde{G}(x, y) \leq c' m^4. \end{aligned}$$

□

Remark 3.4. *The same argument works if we consider $\mathbb{E}(H_{Q(x_1, \lambda m)}(\beta)^p | \alpha)$, $p = 1, 2$, for any $\lambda \in (0, \frac{1}{2})$.*

We now turn to the harder problem of obtaining a lower bound on $\mathbb{E}H_A(\beta)$, and begin with a boundary Harnack inequality which extends [Mas, Proposition 3.5] to higher dimensions. See [BK] for further extensions. In what follows $\mathcal{R}_m = \mathbb{H}_m \cap Q_m$ is the ‘right hand face’ of Q_m .

Lemma 3.5. *Assume $d \geq 1$. Let \mathcal{K} be an arbitrary nonempty subset of $[-m+1, 0] \times [-m+1, m-1]^{d-1}$. For all $m \geq 1$ and all \mathcal{K} we have*

$$\mathbb{P}^0(S(\tau_{Q(0, m-1)}) \in \mathcal{R}_m \mid \tau_{Q(0, m-1)} < T_{\mathcal{K}}^+) \geq (2d)^{-1}. \quad (3.8)$$

Proof. Let $h(z) = \mathbb{P}^z(S_{\tau_{Q(0, m-1)}} \in \mathcal{R}_m)$, $z \in Q(0, m-1)$. By symmetry we have $h(0) = 1/2d$. We first show that

$$h(z) \leq h(0) \text{ for all } z \in ([-m+1, 0] \times [-m+1, m-1]^{d-1}) \cap \mathbb{Z}^d. \quad (3.9)$$

Let $z' = (0, z_2, \dots, z_d)$. Let X^z and $X^{z'}$ be simple random walks with starting points z and z' respectively; we have $h(z) = \mathbb{P}(X_{\tau_A}^z \in \mathcal{R}_n)$, with a similar expression for $h(z')$. We couple these random walks by taking $X^z = z + S$, $X^{z'} = z' + S$, where S is a SRW with $S_0 = 0$. Then $\{X_{\tau_A}^z \in \mathcal{R}_n\} \subset \{X_{\tau_A}^{z'} \in \mathcal{R}_n\}$, and so $h(z) \leq h(z')$.

To prove that $h(z') \leq h(0)$ we use a coupling of continuous time random walks Y, Y' with $Y_0 = 0, Y'_0 = z'$; these have the same exit distribution as the discrete time walk S . Recall that π_j is the projection onto the j th coordinate axis, so that $\pi_j(Y_t)$ gives the j th coordinate of Y_t ; each coordinate is a continuous time simple random walk (run at rate $1/d$) on \mathbb{Z} .

The coupling is as follows. If at time t we have $\pi_j(Y_t) = \pi_j(Y'_t)$ then we run the two j th coordinate processes together, so $\pi_j(Y_{t+s}) = \pi_j(Y'_{t+s})$ for all $s \geq 0$.

Note that we have $|\pi_j(Y_t)| \leq |\pi_j(Y'_t)|$ when $t = 0$; the coupling will preserve this inequality for all $t \geq 0$. If $|\pi_j(Y_t) - \pi_j(Y'_t)| \geq 2$ then we use reflection coupling, so that $\pi_j(Y_t)$ and $\pi_j(Y'_t)$ jump at the same time, and in opposite directions. Finally, suppose that $|\pi_j(Y_t) - \pi_j(Y'_t)| = 1$, and let $a = \pi_j(Y_t)$, $a+1 = \pi_j(Y'_t)$. We take three independent Poisson processes on \mathbb{R}_+ , $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$; each with rate $1/2d$, and make the first jump of either $\pi_j(Y)$ or $\pi_j(Y')$ after time t to be at time $t+T$, where T is the first point in $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$. If $T \in \mathcal{P}_1$ we set $\pi_j(Y_{t+T}) = a-1$, $\pi_j(Y'_{t+T}) = a+2$. If $T \in \mathcal{P}_2$ then we set $\pi_j(Y_{t+T}) = a+1$, $\pi_j(Y'_{t+T}) = a+1$, and if $T \in \mathcal{P}_3$ then $\pi_j(Y_{t+T}) = a$, $\pi_j(Y'_{t+T}) = a$. With this coupling we have $\{Y'_{\tau_A(Y')} \in \mathcal{R}_n\} \subset \{Y_{\tau_A(Y)} \in \mathcal{R}_n\}$, and so $h(z') \leq h(0)$.

Stopping the bounded martingale $h(S(k))$ at $\tau_{Q(0, m-1)} \wedge T_{\mathcal{K}}$, and using (3.9) we get

$$\begin{aligned} h(0) &= \sum_{y \in \mathcal{K}} h(y) \mathbb{P}^0(S(\tau_{Q(0, m-1)} \wedge T_{\mathcal{K}}^+) = y) + \mathbb{P}^0(\tau_{Q(0, m-1)} < T_{\mathcal{K}}^+, S(\tau_{Q(0, m-1)}) \in \mathcal{R}_m) \\ &\leq h(0) \mathbb{P}(\tau_{Q(0, m-1)} > T_{\mathcal{K}}^+) + \mathbb{P}(\tau_{Q(0, m-1)} < T_{\mathcal{K}}^+, S(\tau_{Q(0, m-1)}) \in \mathcal{R}_m). \end{aligned}$$

Rearranging gives the statement of the lemma. \square

We will also need two extensions of Lemma 3.5 that we prove next.

Lemma 3.6. *Assume $d \geq 3$. Let $N \geq 1$ and $Q_{4N} \subset D \subset \mathbb{Z}^d$. Let $8 \leq m \leq N/2$ and $n \leq N$. Suppose that \mathcal{K} is an arbitrary nonempty subset of Q_n , and $x_0 \in \mathcal{K} \cap \mathbb{H}_n$. Let $z_0 = x_0 + me_1$. There exists a constant $c = c(d) > 0$ such that*

$$\mathbb{P}^{z_0}(T_{Q(x_0, m/2)} > \tau_D \mid T_{\mathcal{K}} > \tau_D) \geq c. \quad (3.10)$$

Proof. It is easy to see that the statement holds when $m \geq n/8$, since then $\mathbb{P}^{z_0}(T_{Q_{n+m/2}} > \tau_D) \geq \mathbb{P}^{z_0}(T_{Q_{n+m/2}} = \infty) \geq c$. Henceforth we assume that $m < n/8$.

Let $f(z) = \mathbb{P}^z(T_{\mathcal{K}} > \tau_D)$ and $g(z) = \mathbb{P}^z(T_{\mathcal{K}} \wedge T_{Q(x_0, m/2)} > \tau_D)$, so that we have to prove $f(z_0) \leq Cg(z_0)$. Let $z_1 = x_0 + 8me_1$. Due to the Harnack principle, it is sufficient to show that $f(z_1) \leq Cg(z_1)$.

We first show that for all $y \in \partial Q(x_0, 8m)$ we have $g(y) \leq Cg(z_1)$. Let us write \mathbb{H} for the hyperplane \mathbb{H}_{n+4m} , and \mathbb{H}' for the hyperplane \mathbb{H}_{n+2m} . Observe that \mathbb{H} and \mathbb{H}' are both disjoint from $\mathcal{K} \cup Q(x_0, m/2)$, and they both separate $\mathcal{K} \cup Q(x_0, m/2)$ from z_1 .

If $y \in \partial Q(x_0, 8m)$ lies on the same side of \mathbb{H}' as z_1 , then y is at least distance m from $\mathcal{K} \cup Q(x_0, m/2)$, and this is comparable to the distance between y and z_1 . Hence for such y , the Harnack principle implies $g(y) \leq Cg(z_1)$.

Suppose now that \mathbb{H}' separates y from z_1 . Let $Q^{(1)}$ and $Q^{(2)}$ be cubes that are both translates of Q_{2N} , such that:

- (i) the right hand face of $Q^{(1)}$ and the left hand face of $Q^{(2)}$ coincide;
- (ii) the common set $\mathcal{R} = Q^{(1)} \cap Q^{(2)}$, is contained in \mathbb{H} ;
- (iii) the center of \mathcal{R} (viewed as a $(d-1)$ -dimensional cube), is the point $x_0 + 4me_1$.

Since $g(S(n \wedge \tau_{Q^{(1)}}))$ is a submartingale under \mathbb{P}^y , we have

$$g(y) \leq \mathbb{E}^y(g(\tau_{Q^{(1)}})) = \sum_{w \in \partial Q^{(1)} \setminus \mathcal{R}} g(w) \mathbb{P}^y(S(\tau_{Q^{(1)}}) = w) + \sum_{u \in \mathcal{R}} g(u) \mathbb{P}^y(S(\tau_{Q^{(1)}}) = u). \quad (3.11)$$

Since $g(S(n \wedge \tau_{Q^{(2)}}))$ is a martingale under \mathbb{P}^{z_1} , we also have

$$g(z_1) = \mathbb{E}^{z_1}(g(\tau_{Q^{(2)}})) = \sum_{w' \in \partial Q^{(2)} \setminus \mathcal{R}} g(w') \mathbb{P}^{z_1}(S(\tau_{Q^{(2)}}) = w') + \sum_{u \in \mathcal{R}} g(u) \mathbb{P}^{z_1}(S(\tau_{Q^{(2)}}) = u). \quad (3.12)$$

The mirror symmetry between $Q^{(1)}$ and $Q^{(2)}$, as well as the Harnack principle implies that

$$\begin{aligned} \mathbb{P}^y(S(\tau_{Q^{(1)}}) = u) &\leq C \mathbb{P}^{z_1}(S(\tau_{Q^{(2)}}) = u) \\ \mathbb{P}^y(S(\tau_{Q^{(1)}}) = w) &\leq C \mathbb{P}^{z_1}(S(\tau_{Q^{(2)}}) = w'), \end{aligned}$$

where w' is the mirror image of $w \in \partial Q^{(1)} \setminus \mathcal{R}$ in the hyperplane \mathbb{H} . We also have $g(w) \leq 1$, $w \in \partial Q^{(1)} \setminus \mathcal{R}$, and $g(w') \geq c$, $w' \in \partial Q^{(2)}$. These observations and (3.11) and (3.12) together imply $g(y) \leq Cg(z_1)$.

We now show the desired inequality $f(z_1) \leq Cg(z_1)$. Let $1 \leq R < \infty$ denote the random variable that counts the number of times S^{z_1} makes a crossing from $\partial Q(x_0, 8m)$ to $Q(x_0, m/2)$ before $T_{\mathcal{K}} \wedge \tau_D$. We have

$$\mathbb{P}^{z_1}(R \geq \ell) \leq \left(\max_{y \in \partial Q(x_0, 8m)} \mathbb{P}^y(T_{Q(x_0, m/2)} < \infty) \right)^\ell \leq \gamma^\ell$$

with some $0 < \gamma = \gamma(d) < 1$.

Using the strong Markov property at the time when the ℓ -th crossing has occurred, we can write

$$\begin{aligned}
f(z_1) &= \sum_{\ell=0}^{\infty} \mathbb{P}^{z_1}(R = \ell, T_K > \tau_D) = g(z_1) + \sum_{\ell=1}^{\infty} \mathbb{P}^{z_1}(R = \ell, T_K > \tau_D) \\
&\leq g(z_1) + \sum_{\ell=1}^{\infty} \mathbb{P}^{z_1}(R \geq \ell) \max_{z \in Q(x_0, m/2)} \mathbb{P}^z((T_{Q(x_0, m/2)} \wedge T_K) \circ \Theta_{\tau_{Q(x_0, 8m)}} > \tau_D) \\
&\leq g(z_1) + \sum_{\ell=1}^{\infty} \gamma^\ell \max_{y \in \partial Q(x_0, 8m)} g(y) \\
&\leq g(z_1) + Cg(z_1).
\end{aligned}$$

This completes the proof of the Lemma.

Lemma 3.7. *Assume $d \geq 3$. Let $N \geq 1$ and $Q_{4N} \subset D \subset \mathbb{Z}^d$. Let $8 \leq m \leq N/2$ and $n \leq N$. Suppose that K is an arbitrary nonempty subset of Q_n , and $x_0 \in K \cap \mathbb{H}_n$. Let $\mathcal{R}_{n,m}$ denote the right hand face of $Q(x_0, m)$. There exists a constant $c = c(d) > 0$ such that*

$$\mathbb{P}^{x_0}(S(\tau_{Q(x_0, m)}) \in \mathcal{R}_{n,m} \mid T_K^+ > \tau_D) \geq c. \quad (3.13)$$

Proof. Let $K_0 = K \cap Q(x_0, 2m)$ and $K_1 = K \setminus K_0 = K \setminus Q(x_0, 2m)$. Due to the boundary Harnack inequality, Lemma 3.5, we have

$$\mathbb{P}^{x_0}(S(\tau_{Q(x_0, m)}) \in \mathcal{R}_{n,m} \mid T_K^+ > \tau_{Q(x_0, m)}) \geq (2d)^{-1}. \quad (3.14)$$

Let Z denote the process that is S conditioned on $T_{K_1} > \tau_D$. Then (3.14) and an application of the Harnack principle implies that

$$\mathbb{P}^{x_0}(Z(\tau_{Q(x_0, m)}) \in \mathcal{R}_{n,m} \mid T_K^+[Z] > \tau_{Q(x_0, m)}[Z]) \geq c. \quad (3.15)$$

This in turn implies that

$$\begin{aligned}
&\mathbb{P}^{x_0}(S(\tau_{Q(x_0, m)}) \in \mathcal{R}_{n,m}, T_K^+ > \tau_{Q(x_0, m)}, T_{K_1} > \tau_D) \\
&\geq c \mathbb{P}^{x_0}(T_K^+ > \tau_{Q(x_0, m)}, T_{K_1} > \tau_D) \\
&\geq c \mathbb{P}^{x_0}(T_K^+ > \tau_D).
\end{aligned} \quad (3.16)$$

Let $z_0 = x_0 + 4me_1$. Using the Harnack principle, the left hand side of (3.16) can be bounded from above by

$$\begin{aligned}
&\mathbb{P}^{x_0}(S(\tau_{Q(x_0, m)}) \in \mathcal{R}_{n,m}, T_K^+ > \tau_{Q(x_0, m)}) \max_{z \in \mathcal{R}_{n,m}} \mathbb{P}^z(T_{K_1} > \tau_D) \\
&\leq C \mathbb{P}^{x_0}(S(\tau_{Q(x_0, m)}) \in \mathcal{R}_{n,m}, T_K^+ > \tau_{Q(x_0, m)}) \mathbb{P}^{z_0}(T_{K_1} > \tau_D).
\end{aligned} \quad (3.17)$$

An application of Lemma 3.6 (with $2m$ playing the role of $m/2$) shows that

$$\mathbb{P}^{z_0}(T_{K_1} > \tau_D) \leq C \mathbb{P}^{z_0}(T_{K_1 \cup Q(x_0, 2m)} > \tau_D) \leq C \mathbb{P}^{z_0}(T_K > \tau_D).$$

Substituting this into (3.17), and using the Harnack principle again, we get that the right hand side of (3.17) is bounded above by

$$\begin{aligned}
& C \mathbb{P}^{x_0}(S(\tau_{Q(x_0, m)}) \in \mathcal{R}_{n, m}, T_{\mathcal{K}}^+ > \tau_{Q(x_0, m)}) \mathbb{P}^{z_0}(T_{\mathcal{K}} > \tau_D) \\
& \leq C \mathbb{P}^{x_0}(S(\tau_{Q(x_0, m)}) \in \mathcal{R}_{n, m}, T_{\mathcal{K}}^+ > \tau_{Q(x_0, m)}) \min_{z \in \mathcal{R}_{n, m}} \mathbb{P}^z(T_{\mathcal{K}} > \tau_D) \\
& \leq C \mathbb{P}^{x_0}(S(\tau_{Q(x_0, m)}) \in \mathcal{R}_{n, m}, T_{\mathcal{K}}^+ > \tau_D).
\end{aligned} \tag{3.18}$$

The inequalities (3.16), (3.17) and (3.18) together imply the claim of the Lemma.

We now return to the task of giving a lower bound for $\mathbb{E}(H_A(\beta))$. We will need the following lower bound on \tilde{G} .

Lemma 3.8. *Assume $d \geq 3$. Let $z \in A$. Then*

$$\tilde{G}_D(x_0, z) \geq cm^{2-d}.$$

Proof. This uses the extension of the boundary Harnack inequality, Lemma 3.7. Let V_z be the number of hits on z by \tilde{X} before $\tilde{\tau}_D$. Let $\tilde{T} = \tilde{T}_{\partial_i Q(x_0, m/8)}$. Note that $Q(x_0, m/8)$ and A^* intersect on one of the faces of $Q(x_0, m/8)$. Then since $\tilde{T} < \tilde{\tau}_D$,

$$\begin{aligned}
\tilde{G}_D(x_0, z) &= \mathbb{E}^{x_0} V_z = \mathbb{E}^{x_0} \left(\mathbb{E}^{\tilde{X}_{\tilde{T}}} V_z \right) \geq \mathbb{E}^{x_0} \left(1_{(\tilde{X}_{\tilde{T}} \in A^*)} \min_{y \in \partial_i A^*} \mathbb{E}^y V_z \right) \\
&= \mathbb{P}^{x_0}(\tilde{X}_{\tilde{T}} \in A^*) \min_{y \in \partial_i A^*} \tilde{G}_D(y, z).
\end{aligned}$$

Using (3.3) and (3.4) we have $\tilde{G}_D(y, z) \asymp G_D(y, z) \asymp m^{2-d}$ if $y \in \partial_i A^*$. Let $T = T_{\partial_i Q(x_0, m/8)}$ (for S). Lemma 3.7 implies

$$\mathbb{P}^{x_0}(\tilde{X}_{\tilde{T}} \in A^*) = \mathbb{P}^{x_0}(S_T \in A^* | T_{\alpha}^+ > \tau_D) \geq c,$$

and the Lemma follows. \square

The key estimate is the following.

Lemma 3.9. *Assume $d \geq 5$. Then*

$$\mathbb{E}(H_A(\beta)|\alpha) \geq cm^2. \tag{3.19}$$

Proof. It is enough to prove that if $z \in A$ then

$$\mathbb{P}(z \in \beta|\alpha) \geq cm^{2-d}. \tag{3.20}$$

Let Y be \tilde{X} conditioned to hit z before $\tilde{T}_{\alpha}^+ \wedge \tilde{\tau}_D$, and let \tilde{X}^z be independent of Y . Let

$$Y' = \mathcal{E}_z^L(\mathcal{E}_{\partial D}^F(Y)),$$

so Y' is the path of Y up to its last hit on z before its first exit from D . Let also $X' = \Theta_1 \mathcal{E}_{\partial D}^F \tilde{X}^z$. (We need to apply Θ_1 since the last point of Y' and the first point of X' are both z .) Then as in Lemma 6.1 of [BM1] we have

$$\mathbb{P}(z \in \beta | \alpha) = \tilde{G}_D(x_0, z) \mathbb{P}(\mathcal{L}Y' \cap X' = \emptyset, \mathcal{L}Y' \subset Q(x_0, m)). \quad (3.21)$$

Due to Lemma 3.8, it remains to show that the probability on the right hand side is bounded away from 0. We will in fact prove the stronger statement:

$$\mathbb{P}(Y' \cap X' = \emptyset, Y' \subset Q(x_0, m)) \geq c > 0. \quad (3.22)$$

This result is not surprising, since two independent SRW in \mathbb{Z}^d (with $d \geq 5$) intersect with probability strictly less than 1.

Let us denote $A_z = Q(z, m/16)$, $B = Q(x_0, m)$ and $B' = Q(x_0, m/16)$. Note that Y' starts at x_0 and ends at z . We decompose Y' into four subpaths, defined below, and give separate estimates for these subpaths that together will imply the lower bound on the probability in (3.22). We define:

$$Y'_1 = \mathcal{E}_{\partial B'}^F(Y') \quad Y'_2 = \mathcal{E}_{\partial A_z}^L(\mathcal{B}_{\partial B'}^F(Y')) \quad Y'_3 = \mathcal{B}_{\partial A_z}^L(Y').$$

That is, Y'_1 ends at the first exit from B' , Y'_3 begins at the last entrance to A_z and Y'_2 is the portion in between. We let $y_1 = Y'_1(|Y'_1|) = Y'_2(0)$ and $y_2 = Y'_2(|Y'_2|) = Y'_3(0)$. We further decompose Y'_2 into the pieces:

$$Y'_{2,1} = \mathcal{E}_{y_2}^F(Y'_2) \quad Y'_{2,2} = \mathcal{B}_{y_2}^F(Y'_2).$$

That is, $Y'_{2,1}$ is the piece from y_1 to the first hit on y_2 , and $Y'_{2,2}$ is the remaining loop at y_2 . Observe that conditional on y_1 and y_2 , the paths $Y'_1, Y'_{2,1}, Y'_{2,2}, Y'_3$ are independent. We now state our estimates for each piece. Our notation will assume that $x_0 \in \mathbb{H}_n$; trivial modification can be made when this is not the case.

Claim 1. There is constant probability that Y'_1 exits B' on the right hand face. That is, we have $\mathbb{P}(y_1 \in \mathcal{R}_{n,m/16}) \geq c > 0$, where $\mathcal{R}_{n,m/16} = \mathbb{H}_{n+m/16} \cap Q(x_0, m/16)$.

Proof of Claim 1. Using Lemma 3.7 we have

$$\begin{aligned} \mathbb{P}(y_1 \in \mathcal{R}_{n,m/16}) &= \frac{\mathbb{P}^{x_0}(\tilde{X}(\tilde{\tau}_{B'}) \in \mathcal{R}_{n,m/16}, \tilde{T}_z < \tilde{\tau}_D)}{\mathbb{P}^{x_0}(\tilde{T}_z < \tilde{\tau}_D)} \\ &\geq \frac{\tilde{G}_D(z, z)}{\tilde{G}_D(x_0, z)} \mathbb{P}^{x_0}(\tilde{X}(\tilde{\tau}_{B'}) \in \mathcal{R}_{n,m/16}) \min_{w \in \mathcal{R}_{n,m/16}} \mathbb{P}^w(\tilde{T}_z < \tau_D) \\ &\geq c \min_{w \in \mathcal{R}_{n,m/16}} \frac{\tilde{G}_D(w, z)}{\tilde{G}_D(x_0, z)} \geq c. \end{aligned}$$

In the next three claims we will use the notation $B'' = x_0 + ([0, z_1 + m/32] \times [-m, m]^{d-1}) \cap \mathbb{Z}^d$.

Claim 2. There is constant probability that the following six events occur:

- (i) Y'_3 starts on the left hand face of A_z ;
- (ii) $Y'_3 \subset z + ([-m/16, m/32] \times [-m/16, m/16]^{d-1}) \cap \mathbb{Z}^d$;
- (iii) X' exits A_z on the right hand face;
- (iv) $X' \cap A_z \subset z + ([-m/32, m/16] \times [-m/16, m/16]^{d-1}) \cap \mathbb{Z}^d$;
- (v) $Y'_3 \cap (X' \cap A_z) = \emptyset$;
- (vi) $\mathcal{B}_{\partial A_z}^F(X')$ is disjoint from B'' .

Proof of Claim 2. Let \tilde{S}^z be the process defined as S^z conditioned to hit on x_0 before $T_{\alpha \setminus \{x_0\}} \wedge \tau_D$. The time-reversal of Y' has the law of \tilde{S}^z . Therefore, the time-reversal of Y'_3 has the law of $\mathcal{E}_{\partial A_z}^F(\tilde{S}^z)$. The proof of Lemma 3.2 (Separation Lemma), shows that for independent simple random walks S^z and S'^z there is probability $\geq c > 0$ that the analogues of the events (i)–(v) all hold. An application of the Harnack principle then shows that in fact (i)–(v) hold with constant probability.

It is left to show that conditionally on (i)–(v), we also have (vi) with constant probability. Since X' is S conditioned on $T_\alpha > \tau_D$, this can be proved in the same way as Lemma 3.6. For this we merely have to replace $Q(x_0, m/2)$ in that lemma by B'' , and make straightforward adjustments. Hence Claim 2 follows.

Claim 3. Conditional on y_1 being in the right hand face of B' and y_2 being in the left hand face of A_z , there is constant probability that $Y'_{2,1} \subset B''$.

Proof of Claim 3. Condition on y_1 and y_2 . Then $Y'_{2,1}$ has the law of S^{y_1} conditioned to hit on y_2 before $T_\alpha \wedge \tau_D$ (stopped at the first hit on y_2). Since y_1 and y_2 are at least distance cm from the boundary of B'' , such a path has constant probability to stay inside B'' . (One way to see this is to use an argument similar to that of Lemma 3.6, where we let R count the number of crossings by the walk from $Q(z, m/64)$ to $\partial B''$ before time $T_z \wedge T_\alpha \wedge \tau_D$.) Hence the claim follows.

Claim 4. Conditional on y_2 being in the left hand face of A_z , there is constant probability that $Y'_{2,2} \subset B''$.

Proof of Claim 4. Condition on y_2 . The probability that $Y'_{2,2}$ consists of a single point is $G_{D \setminus \alpha}(y_2, y_2)^{-1} \geq G(y_2, y_2)^{-1} \geq c > 0$.

When all the events in Claims 1–4 occur, the event in (3.22) occurs. Hence the Lemma follows.

An application of Lemmas 3.3 and 3.9 and the one-sided Chebyshev inequality give the following corollary.

Corollary 3.10. *When $d \geq 5$, there exists a constant $c_0 > 0$ such that*

$$\mathbb{P}(H_A(\beta) \geq c_0 m^2 | \alpha) \geq c_0.$$

Proposition 3.11. *Assume $d \geq 5$. Let $N \geq 1$ and $Q_{4N} \subset D \subset \mathbb{Z}^d$. Let $L = \mathcal{L}\mathcal{E}_{\partial D}^F S$ be a loop erased walk from 0 to ∂D , and $M_N = |\mathcal{E}_{\partial_i Q_N}^F L|$ be the number of steps in L until its first hit on $\partial_i Q_N$. Then for all $\lambda > 0$ we have*

$$\mathbb{P}(M_N < \lambda N^2) \leq C \exp(-c\lambda^{-1}). \quad (3.23)$$

Proof. Suppose $k \geq 1$ and $m \geq 4$ such that $N/2 \leq km < N - m$. For $j = 1, \dots, k$ let

$$\alpha_j = \mathcal{E}_{\partial_i Q(0, jm)}^F L, \quad \mathcal{F}_j = \sigma(\alpha_j).$$

Let $Y_j = \alpha_j(|\alpha_j|)$ be the last point in α_j , and

$$\beta_j = \mathcal{E}_{\partial_i Q(Y_j, m)}^F (\mathcal{B}_{\partial_i Q(0, jm)}^F L)$$

be the path L between Y_j and its first hit after Y_j on $\partial_i(Y_j, m)$. We have

$$M_N \geq \sum_{i=1}^k |\beta_j|,$$

Let $G_j = \{|\beta_j| < c_0 m^2\}$; then by Corollary 3.10

$$\mathbb{P}(G_j | \mathcal{F}_j) \leq 1 - c_0.$$

Therefore, M_N stochastically dominates a sum of k independent random variables that take the values $c_0 m^2$ and 0 with probabilities c_0 and $1 - c_0$, respectively. Hence

$$\mathbb{P}(M_N \leq (1/2)kc_0^2 m^2) \leq C \exp(-ck).$$

We now take $k \asymp \lambda^{-1}$ and $m \asymp \lambda N$ and we obtain (3.23). \square

In the following theorem, we obtain a lower bound on the length of paths in the USF. We define the event:

$$F(y, x, n) = \{T_x[S^y] < \infty \text{ and } |\mathcal{L}\mathcal{E}_x^F(S^y)| \leq n\}. \quad (3.24)$$

Theorem 3.12. *For every $x, y \in \mathbb{Z}^d$ we have*

$$\mathbb{P}(F(y, x, n)) \leq C(1 + |x - y|)^{2-d} \exp\left[-c \frac{|x - y|^2}{n}\right]. \quad (3.25)$$

Proof. For notational convenience, we assume $y = 0$ (otherwise translate x, y by $-y$). If $|x|^2/n \leq 1$ then the term in the exponential in (3.25) is of order 1, so

$$\mathbb{P}(F(0, x, n)) \leq \mathbb{P}(T_x < \infty) \leq (1 + |x|)^{2-d} \leq e^c (1 + |x|)^{2-d} e^{-c|x|^2/n}.$$

Now assume $|x|^2 > n$, and let $N = \lfloor |x| \rfloor / 4$, and $Q = Q(0, N)$. Let X' be S conditioned on $\{T_x < \infty\}$. Then if $h(z) = \mathbb{P}^z(T_x[S] < \infty)$, we have $h(z) \asymp N^{2-d}$ on $Q(0, N)$, and thus the processes S and X' have comparable laws inside $Q(0, N)$. The explicit law of a section of the loop erased random path given in [Law99] (see also (5) in [Mas]) then implies that the loop erasures of S and X' also have comparable laws inside Q .

Let

$$F_1(x, n) = \{|\mathcal{E}_{\partial_i Q}^F(\mathcal{L}\mathcal{E}_x^F S)| \leq n, T_x < \infty\}. \quad (3.26)$$

Thus $F(0, x, n) \subset F_1(x, n)$. Then

$$\begin{aligned}
\mathbb{P}(F(0, x, n)) &\leq \mathbb{P}(F_1(x, n)) \\
&= \mathbb{P}(|\mathcal{E}_{\partial_i Q}^F \mathcal{L}(\mathcal{E}_x^F S)| \leq n | T_x < \infty) \mathbb{P}(T_x < \infty) \\
&\leq C|x|^{2-d} \mathbb{P}(|\mathcal{E}_{\partial_i Q}^F \mathcal{L}(\mathcal{E}_x^F X')| \leq n) \\
&\leq C|x|^{2-d} \mathbb{P}(|\mathcal{E}_{\partial_i Q}^F \mathcal{L}(\mathcal{E}_x^F S)| \leq n).
\end{aligned}$$

Taking $n = \lambda N^2$, so that $\lambda^{-1} \geq c|x|^2 n^{-1}$, and using Proposition 3.11 completes the proof. \square

4 Upper bound on $|B_{\mathcal{U}}(0, n)|$

Recall that $\mathcal{U}(x)$ is the component of the USF containing $x \in \mathbb{Z}^d$. It is well-known [Pem91, Theorem 4.2] that for $d \geq 5$ and $x \neq y \in \mathbb{Z}^d$ we have

$$c|x - y|^{4-d} \leq \mathbb{P}(y \in \mathcal{U}(x)) \leq C|x - y|^{4-d}. \quad (4.1)$$

A corollary of this bound is that the volume of $\mathcal{U}_0 \cap B(r)$ grows as r^4 in expectation. Our main result in the previous section, Theorem 3.12, is a variant of the upper bound in (4.1) that gives control over the length of the path connecting x and y . Since that bound was formulated in terms of a single LERW, the exponent $4 - d$ changes to $2 - d$. In this section we extend Theorem 3.12 to control the volume of balls in the intrinsic metric.

Theorem 4.1. *Assume $d \geq 5$, and let $\mathcal{U} = \mathcal{U}_{\mathbb{Z}^d}$. There exists a constant C_1 such that for all $k \geq 0$ we have*

$$\mathbb{E}(|B_{\mathcal{U}}(0, n)|^k) \leq C_1^k k! n^{2k}. \quad (4.2)$$

Hence there are constants $c_1 > 0$ and C_2 such that

$$\mathbb{P}(|B_{\mathcal{U}}(0, n)| \geq \lambda n^2) \leq C_2 e^{-c_1 \lambda}, \quad \lambda > 0, n \geq 1. \quad (4.3)$$

Proof. The bound (4.3) follows easily from (4.2) using Markov's inequality and the power series for e^x .

We prove (4.2) by induction on k . The case $k = 0$ holds trivially. We fix $k \geq 1$ and $y_1, \dots, y_k \in \mathbb{Z}^d$, and estimate the probability

$$\mathbb{P}(y_1, \dots, y_k \in B_{\mathcal{U}}(0, n)).$$

This can be done similarly to the “tree-graph inequalities” known in percolation [AN]. To facilitate notation, we write $y_0 = 0$. On the event $y_1, \dots, y_k \in \mathcal{U}_0$ consider the minimal subtree $T(y_0, \dots, y_k) \subset \mathcal{U}_0$ that contains the vertices y_0, \dots, y_k . This tree is finite. Since \mathcal{U}_0 has one end [BLPS], [LP], there is a unique infinite path in \mathcal{U}_0 , whose only vertex in $T(y_0, \dots, y_k)$ is its starting vertex. Let us write $T(y_0, \dots, y_k, \infty)$ for the infinite subtree of \mathcal{U}_0 obtained by adding this infinite path to $T(y_0, \dots, y_k)$.

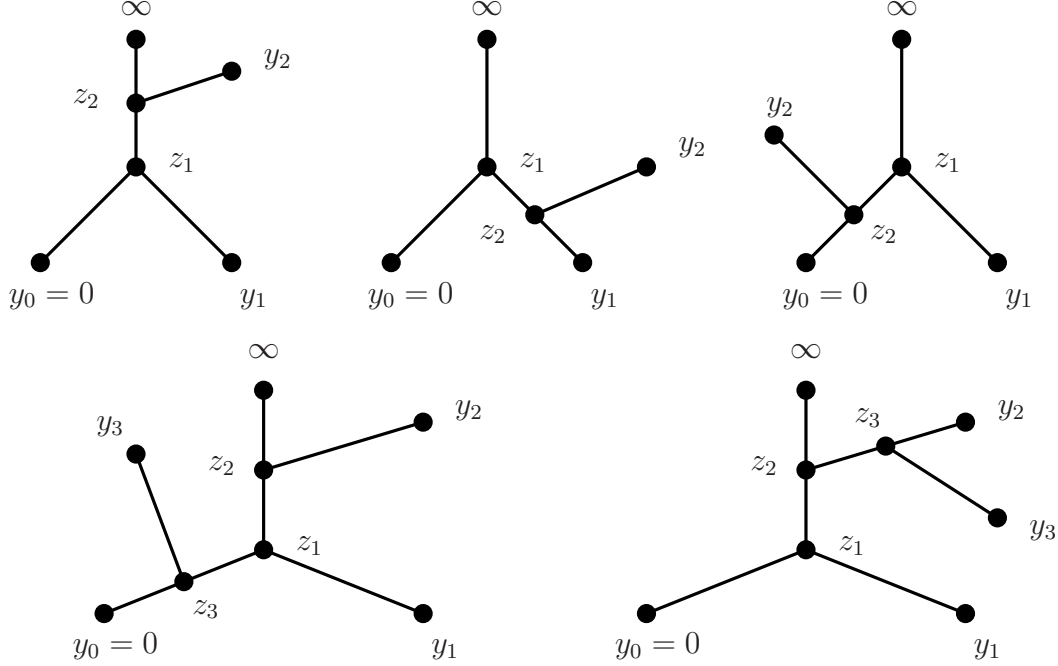


Figure 2: All three labelled tree graphs with $k = 2$, and two of the five possible labelled tree graphs with $k = 3$.

Now let us consider the “topology” of $T(y_0, \dots, y_k, \infty)$. In the case $k = 1$, it is easy to see that there exists a vertex $z_1 \in T(y_0, y_1, \infty)$ such that the paths $T(y_0, z_1)$, $T(y_1, z_1)$ and $T(z_1, \infty)$ (some of which may degenerate to a single vertex) are edge-disjoint. In the general case $k \geq 1$, we have k “branch points” z_1, \dots, z_k . We use a fixed rule for indexing the z_i ’s, in requiring that for every $i \geq 1$ the path $T(y_i, z_i)$ is edge-disjoint from $T(y_0, \dots, y_{i-1}, \infty)$. See Figure 2.

We can formalize the construction via the following recursive procedure. Let $\mathcal{T}(0)$ denote the set containing the unique tree with vertex set $\{0, \infty\}$. Assume that the collection $\mathcal{T}(k-1)$ of trees with vertex set $\{0, \dots, k-1\} \cup \{\infty\} \cup \{\bar{1}, \dots, \bar{k-1}\}$ has been defined for some $k \geq 1$. Let $\mathcal{T}(k)$ denote the collection of trees with vertex set $\{0, \dots, k\} \cup \{\infty\} \cup \{\bar{1}, \dots, \bar{k}\}$ that can be obtained in the following way. Pick some $\tau' \in \mathcal{T}(k-1)$, and pick one of the edges of τ' . Split this edge into two by introducing a new vertex \bar{k} on the edge, and add the new edge $\{k, \bar{k}\}$ to τ' . It is easy to see that any $\tau \in \mathcal{T}(k)$ has the following properties (see Figure 2):

- (i) $\deg_\tau(\infty) = 1 = \deg_\tau(y_i)$, $i = 0, \dots, k$.
- (ii) $\deg_\tau(\bar{i}) = 3$, $i = 1, \dots, k$.

With the above definitions, the event $\{y_1, \dots, y_k \in \mathcal{U}_0\}$ implies that there exist $z_1, \dots, z_k \in T(y_0, \dots, y_k, \infty)$ and $\tau \in \mathcal{T}(k)$ such that $T(y_0, \dots, y_k, \infty)$ is the edge-disjoint union of paths $T(\varphi(r), \varphi(s))$, where $\{r, s\} \in E(\tau)$, and $\varphi : V(\tau) \rightarrow \mathbb{Z}^d \cup \{\infty\}$ is defined

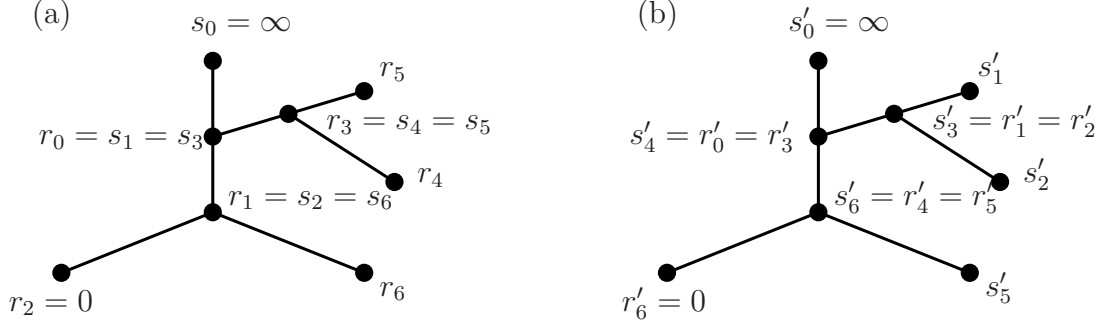


Figure 3: (a) A possible enumeration of edges for the application of Wilson's method. (b) A possible enumeration of edges for performing the summations using (4.8) in the order $j = 1, 2, \dots, 2k$. Summing over the spatial location $\varphi(s'_1)$ eliminates the factor involving the edge $\{s'_1, r'_1\}$. Following this, it is possible to sum over $\varphi(s'_2)$, etc.

by

$$\begin{cases} \varphi(i) = y_i & i = 0, \dots, k; \\ \varphi(\infty) = \infty; \\ \varphi(\bar{i}) = z_i & i = 1, \dots, k. \end{cases} \quad (4.4)$$

Note that the choice of τ is not unique, due to possible coincidences between the vertices $y_0, \dots, y_k, z_1, \dots, z_k$. We neglect the overcounting resulting from this, for an upper bound.

If the additional restriction $d_{\mathcal{U}}(0, y_i) \leq n$, $i = 1, \dots, k$ is in place, we must also have $d_{\mathcal{U}}(\varphi(r), \varphi(s)) \leq n$ for all $\{r, s\} \in E(\tau)$ such that $r, s \neq \infty$. We define the event

$$\begin{aligned} E(y_1, \dots, y_k, z_1, \dots, z_k, \tau, n) \\ = \left\{ \begin{array}{l} T(y_0, \dots, y_k, \infty) = \cup_{\{r, s\} \in E(\tau)} T(\varphi(r), \varphi(s)) \text{ as} \\ \text{an edge-disjoint union and } d_{\mathcal{U}}(\varphi(r), \varphi(s)) \leq n \\ \text{for all } \{r, s\} \in E(\tau) \text{ such that } r, s \neq \infty \end{array} \right\}. \end{aligned}$$

Considering all possible choices of τ and z_1, \dots, z_k , we get

$$\begin{aligned} \mathbb{E}(|B_{\mathcal{U}}(0, n)|^k) &= \sum_{y_1, \dots, y_k \in \mathbb{Z}^d} \mathbb{P}(y_1, \dots, y_k \in B_{\mathcal{U}}(0, n)) \\ &\leq \sum_{\tau \in \mathcal{T}(k)} \sum_{y_1, \dots, y_k \in \mathbb{Z}^d} \sum_{z_1, \dots, z_k \in \mathbb{Z}^d} \mathbb{P}(E(y_1, \dots, y_k, z_1, \dots, z_k, \tau, n)). \end{aligned}$$

We use Wilson's algorithm [W, LP] to replace the complicated event $E(y_1, \dots)$ by a slightly larger event that is easier to handle. For this, enumerate the edges of τ as

$$\{r_0, s_0\}, \{r_1, s_1\}, \dots, \{r_{2k}, s_{2k}\},$$

where the labelling is chosen in such a way that the following two properties are satisfied (see Figure 3(a)):

- (a) $s_0 = \infty$.

- (b) For every $j = 1, \dots, 2k$, the set of edges $\{\{r_\ell, s_\ell\} : \ell = 0, \dots, j-1\}$ spans a subtree of τ , and s_j is a vertex of this subtree.

Using Wilson's method with random walks started at $\varphi(r_0), \dots, \varphi(r_{2k})$, we see that

$$E(y_1, \dots, y_k, z_1, \dots, z_k, \tau, n) \subset \bigcap_{j=1}^{2k} F(\varphi(s_j), \varphi(r_j), n). \quad (4.5)$$

Here $F(\cdot, \cdot, n)$ are the events defined in (3.24). Importantly, the events on the right hand side are independent. Theorem 3.12 and the inclusion (4.5) imply that

$$\begin{aligned} & \mathbb{P}(E(y_1, \dots, y_k, z_1, \dots, z_k, \tau, n)) \\ & \leq \prod_{j=1}^{2k} C(1 + |\varphi(s_j) - \varphi(r_j)|)^{2-d} \exp \left[-c \frac{|\varphi(s_j) - \varphi(r_j)|^2}{n} \right]. \end{aligned} \quad (4.6)$$

It remains to estimate the sum of the right hand side of (4.6) over all choices of the y_i 's and z_i 's. For this it will be convenient to use a different enumeration of $E(\tau)$. Suppose that

$$\{r'_0, s'_0\}, \{r'_1, s'_1\}, \dots, \{r'_{2k}, s'_{2k}\}$$

satisfies the following properties (see Figure 3(b)).

- (a') $s'_0 = \infty$ and $r'_{2k} = 0$.
- (b') For every $j = 1, \dots, 2k$ the set $\{\{r'_\ell, s'_\ell\} : \ell = j, \dots, 2k\}$ induces a connected subtree of τ , and s'_j is a leaf of this subtree.

For ease of notation, let us write $u_j = \varphi(r'_j)$ and $w_j = \varphi(s'_j)$. With the new enumeration the right hand side of (4.6) takes the following form:

$$\begin{aligned} & \mathbb{P}(E(y_1, \dots, y_k, z_1, \dots, z_k, \tau, n)) \\ & \leq \prod_{j=1}^{2k} C(1 + |w_j - u_j|)^{2-d} \exp \left[-c \frac{|w_j - u_j|^2}{n} \right]. \end{aligned} \quad (4.7)$$

Note again that the w_j 's and u_j 's are z_i 's and y_i 's, determined implicitly by τ . Importantly, property (b') of the enumeration implies that if $w_j = \varphi(s'_j) = z_i$ for some i, j , then the variable z_i does not occur in the product

$$\prod_{\ell=j+1}^{2k} C(1 + |w_\ell - u_\ell|)^{2-d} \exp \left[-c \frac{|w_\ell - u_\ell|^2}{n} \right].$$

Similar considerations apply if $w_j = \varphi(s'_j) = y_i$ for some i, j . The summation over y_1, \dots, y_k and z_1, \dots, z_k can be accomplished by the following lemma.

Lemma 4.2. *For any $u \in \mathbb{Z}^d$, we have*

$$\sum_{w \in \mathbb{Z}^d} (1 + |w - u|)^{2-d} \exp \left[-c \frac{|w - u|^2}{n} \right] \leq Cn. \quad (4.8)$$

We apply Lemma 4.2 successively to the factors with $j = 1, \dots, 2k$ on the right hand side of (4.7). See Figure 3(b) for an example of how the edges of τ are successively removed by the summations. We obtain

$$\mathbb{E}(|B_{\mathcal{U}}(0, n)|^k) \leq \sum_{\tau \in \mathcal{T}(k)} (Cn)^{2k}. \quad (4.9)$$

Since the number of trees in $\mathcal{T}(k)$ is $1 \cdot 3 \cdot \dots \cdot (2k - 1) \leq 2^k k!$, this proves (4.2). \square

Remark 4.3. *The statements of Theorem 4.1 still hold, with essentially the same proof, when $\mathcal{U} = \mathcal{U}_D$, with any $D \subset \mathbb{Z}^d$. Note that \mathcal{U}_0 still has one end. This follows from [LMS, Proposition 3.1], and the fact that the component of 0 under the measure \mathbf{WSF}_o in the domain D is stochastically smaller than it is in \mathbb{Z}^d . Therefore, a decomposition into events $E(y_1, \dots, y_k, z_1, \dots, z_k, n)$ still holds (with $\mathcal{U} = \mathcal{U}_D$), where now all vertices are in D . The inclusion (4.5) still holds, with the events F having the same meaning as before. This allows to bound the summations in exactly the same way as in \mathbb{Z}^d .*

5 Lower bounds on volumes

In this section we return to the setup of Section 3, in order to give a lower bound on the volume of \mathcal{U}_0 . We first estimate the number of vertices of \mathcal{U}_0 in shells $Q_{n+m} \setminus Q_n$. Recall that $Q_N \subset D \subset \mathbb{Z}^d$, and n, m satisfy $16 \leq n < n + m \leq N$, with $m \leq n/8$. We have $L = \mathcal{L}(\mathcal{E}_{D^c}^F(S))$, $\alpha = \mathcal{E}_{\partial_i Q_n}^F L$, and $x_0 \in \partial_i Q_n$ is the endpoint of α . The remaining piece of L is $L' = \mathcal{B}_{\partial_i Q_n}^F L$, and $\beta = \mathcal{E}_{\partial_i Q(x_0, m)}^F L'$. See Figure 4. Recall that when $x_0 \in \mathbb{H}_n$, we defined $A = A(x_0) = Q(x_0 + (m/2)e_1, m/4)$ and $x_1 = x_0 + (m/2)e_1$, with appropriate rotations applied when x_0 was on a different face of Q_n . We will now also need a point $x_2 \in Q_{n+m} \setminus Q_n$ of order m away from A , and further boxes contained in $Q_{n+m} \setminus Q_n$ that we define as follows. If $x_0 \in \mathbb{H}_n$ and the second coordinate of x_0 is negative, let

$$\begin{aligned} x_2 &= x_1 + me_2 \\ A' &= A'(x_0) = Q(x_1 + 2me_2, m/4) \\ A'' &= A''(x_0) = x_1 + [-3m/8, 3m/8] \times [-m, 3m] \times [-m, m]^{d-2} \cap \mathbb{Z}^d. \end{aligned} \quad (5.1)$$

If $x_0 \in \mathbb{H}_n$ and the second coordinate of x_0 is positive, we replace e_2 by $-e_2$ and $[-m, 3m]$ by $[-3m, m]$. If x_0 is on a different face of Q_n , we replace e_1 and e_2 by two other suitable unitvectors.

The key technical estimate is to show that $\beta \cap A$ has capacity of order m^2 with probability bounded away from 0, which we do in the next section.

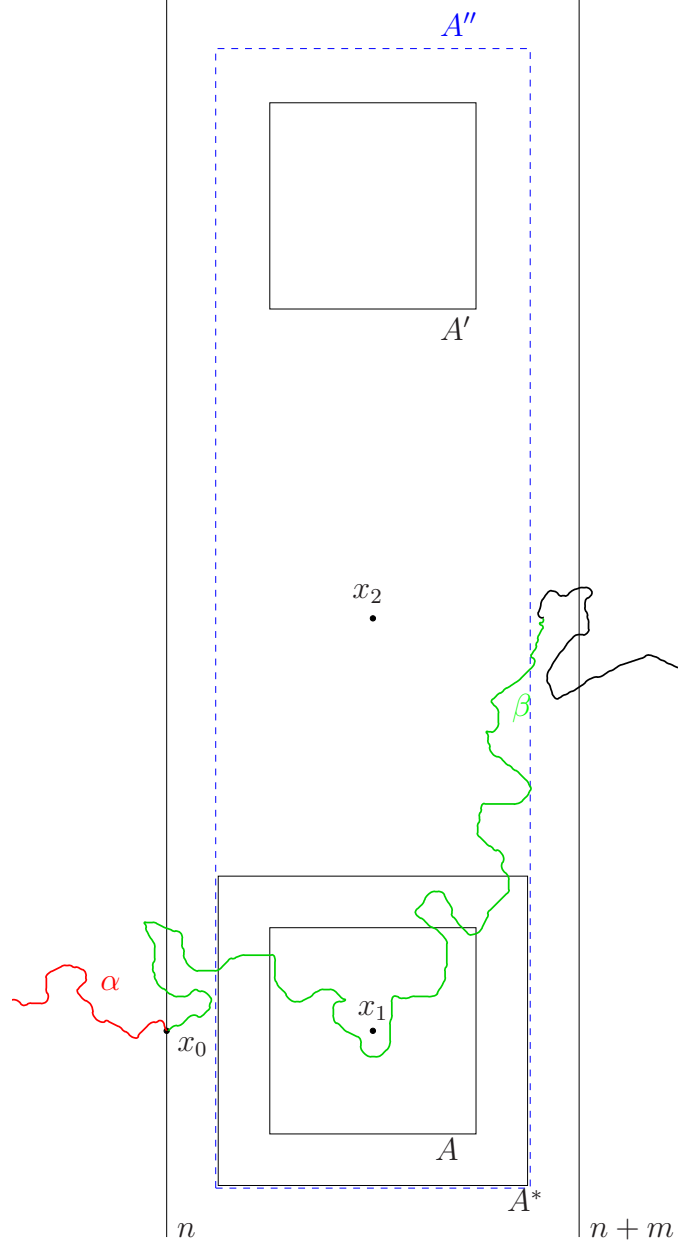


Figure 4: Boxes for the cycle popping argument.

5.1 A capacity estimate

Let S^{x_2} be a random walk with $S^{x_2}(0) = x_2$, independent of S , \tilde{X} , etc.

Proposition 5.1. *Assume $N \geq 1$, $Q_{4N} \subset D \subset \mathbb{Z}^d$, and the setup of Section 3.*

(a) *There exists $c_1 = c_1(d) > 0$ such that*

$$\mathbb{P}(S^{x_2} \text{ hits } (A \cap \beta) \mid \alpha) \geq c_1 m^{4-d}.$$

(b) *We have*

$$\mathbb{P}(cm^2 \leq \text{Cap}(A \cap \beta) \leq C_1 m^2 \mid \alpha) \geq c > 0. \quad (5.2)$$

Proof. (a) For ease of notation, we omit the conditioning on α . Let

$$U := \sum_{z \in A} I[z \in \beta] I[S^{x_2} \text{ hits } z],$$

so that

$$\mathbb{P}(S^{x_2} \text{ hits } (A \cap \beta)) = \mathbb{P}(U > 0).$$

Using Lemma 3.9, we have

$$\mathbb{E}(U) = \sum_{z \in A} \mathbb{P}(z \in \beta) \mathbb{P}(T_z[S^{x_2}] < \infty) \geq cm^d m^{2-d} m^{2-d} = cm^{4-d}.$$

On the other hand,

$$\mathbb{E}(U^2) = \sum_{x, y \in A} \mathbb{P}(x, y \in \beta) \mathbb{P}(T_x[S^{x_2}] < \infty, T_y[S^{x_2}] < \infty). \quad (5.3)$$

Since the process \tilde{X} generating L' must pass through ∂A^* in order for the event $x, y \in \beta$ to occur, we have

$$\begin{aligned} \mathbb{P}(x, y \in \beta) &\leq \max_{z \in \partial A^*} [\tilde{G}_D(z, x) \tilde{G}_D(z, y) + \tilde{G}_D(z, y) \tilde{G}_D(z, x)] \\ &\leq Cm^{2-d} G(x, y). \end{aligned}$$

For the other term in the right hand side of (5.3) we have

$$\begin{aligned} \mathbb{P}(T_x[S^{x_2}] < \infty, T_y[S^{x_2}] < \infty) &\leq [G(x_2, x)G(x, y) + G(x_2, y)G(y, x)] \\ &\leq Cm^{2-d} G(x, y). \end{aligned}$$

Since $d \geq 5$, we have $\sum_{x, y \in A} G(x, y)^2 \leq Cm^d$, which gives $\mathbb{E}(U^2) \leq Cm^{4-d}$.

The Paley-Zygmund inequality then gives

$$\mathbb{P}(S^{x_2} \text{ hits } (A \cap \beta)) = \mathbb{P}(U > 0) \geq \frac{\mathbb{E}(U)^2}{\mathbb{E}(U^2)} \geq cm^{4-d}.$$

(b) Since $\text{Cap}(A \cap \beta) \leq C|A \cap \beta|$, and $m^{2-d}\text{Cap}(A \cap \beta) \asymp \mathbb{P}(T_{A \cap \beta}[S^{x_2}] < \infty \mid \beta)$, combining (a) with Lemma 3.3 gives (b). \square

Assume now, similarly to Proposition 3.11, that $k \geq 1$ and $m \geq 4$ such that $N/2 \leq km < N - m$. Recall that for $j = 1, \dots, k$ we denote $\alpha_j = \mathcal{E}_{\partial_i Q(0, jm)}^F L$. Let $Y_j = \alpha_j(|\alpha_j|)$ be the last point in α_j , and $\beta_j = \mathcal{E}_{\partial_i Q(Y_j, m)}^F(\mathcal{B}_{\partial_i Q(0, jm)}^F L)$ be the path L between Y_j and its first hit after Y_j on $\partial_i(Y_j, m)$. Let $Y_{j,1}$ and $Y_{j,2}$ be the points x_1 and x_2 defined with respect to $x_0 = Y_j$, respectively. Define the following event, measurable with respect to L :

$$G(c_1, c_2, C_1) = \left\{ \begin{array}{l} \text{there are at least } c_2 k \text{ indices } j \text{ with } 1 \leq j \leq k \\ \text{such that } \mathbb{P}(T_{A(Y_j) \cap \beta_j}[S^{Y_{j,2}}] < \infty \mid L) \geq c_1 m^{4-d} \\ \text{and } |Q(Y_j, m) \cap \beta_j| \leq C_2 m^2 \end{array} \right\}. \quad (5.4)$$

Proposition 5.1 and an argument similar to that of Proposition 3.11 gives the following corollary.

Corollary 5.2. *Under the assumptions of Proposition 5.1, there exist $c_1, c_2 > 0$ and C_2 such that we have*

$$\mathbb{P}[G(c_1, c_2, C_2)] \geq 1 - \exp(-ck). \quad (5.5)$$

Remark 5.3. *We note the following minor extension of Corollary 5.2. Assuming still that $Q_{4N} \subset D$, let $w \in \partial D$ be fixed, condition S to exit D at w , and let $L' = \mathcal{L}(\mathcal{E}_{D^c}^F S)$ be the loop-erasure. Masson [Mas] proves that the law of $\mathcal{E}_{Q_N^c}^F L'$ is comparable, up to constants factors, to the law of $\mathcal{E}_{Q_N^c}^F L$. Since the event $G(c_1, c_2, C_2)$ is measurable with respect to $\mathcal{E}_{Q_N^c}^F L$, the statement of the corollary follows also for L' .*

5.2 Lower bound on $|Q_N \cap \mathcal{U}_0|$

We continue with the setup of the previous section. Our argument will use the cycle popping idea of Wilson [W]; see also [LP].

Theorem 5.4. *Assume $N \geq 1$, $Q_{4N} \subset D \subset \mathbb{Z}^d$, and let $\mathcal{U} = \mathcal{U}_D$. There exist constants C, c , such that*

$$\mathbb{P}(|Q_N \cap \mathcal{U}_0| \leq \lambda N^4) \leq C \exp(-c\lambda^{-1/3}).$$

Proof. Condition on L , and assume that the event (5.4) occurs. Let J be the set of indices $1 \leq j \leq k$ (a $\sigma(L)$ -measurable random set) satisfying the requirements in this event. For each $j \in J$, let

$$A'(j) = A'(Y_j) \quad A''(j) = A''(Y_j).$$

The definitions of A' and A'' made in (5.1) ensure that $A''(j)$, $j \in J$ are disjoint.

We will need two coupled collections of stacks. Associate to each $z \in (\cup_{j \in J} A''(j)) \setminus L$ a stack of arrows, and let us call these **Stacks I**. For each $j \in J$ and each $z \in A''(j) \cap L \setminus \beta_j$, pick a new independent stack leaving the rest of the stacks unchanged. Call this second collection of stacks **Stacks II**. In both **Stacks I** and **Stacks II**, and for every $j \in J$, pop all cycles that are entirely contained in $A''(j)$. That is, if a cycle starts in $A''(j)$, but part of it lies outside $A''(j)$, we do not pop it. It is important to note that the order of popping cycles is irrelevant for determining the final configuration on the top of the stacks.

For each $j \in J$, let

$$V_j^I = \left\{ y \in A'(j) : \begin{array}{l} \text{cycle popping using } \mathbf{Stacks I} \\ \text{reveals a path from } y \text{ to } L \end{array} \right\}$$

$$V_j^{II} = \left\{ y \in A'(j) : \begin{array}{l} \text{cycle popping using } \mathbf{Stacks II} \\ \text{reveals a path from } y \text{ to } A''(j) \cap \beta_j \end{array} \right\}$$

Note that $(V_j^I, V_j^{II})_{j \in J}$ are conditionally independent, given L, J .

Lemma 5.5. *We have $V_j^I \supset V_j^{II}$ for all $j \in J$.*

Proof. Let $y \in V_j^{II}$, and consider **Stacks II**. Starting from y , follow the arrows in **Stacks II**, until $A''(j) \cap \beta_j$ is hit. Removing cycles chronologically from this path pops some cycles entirely contained in $A''(j)$, and reveals a path from y to $A''(j) \cap \beta_j$. Now if we follow the arrows in **Stacks I** instead, then the same arrows are used until the first time L is hit. This guarantees that a path from y to L is revealed, that does not leave $A''(j)$, and hence $y \in V_j^I$.

Lemma 5.6. *Assume $d \geq 5$. For some $c_3 > 0$ we have*

$$\mathbb{P}(|V_j^{II}| \geq c_3 m^4 \mid L, j \in J) \geq c > 0.$$

Proof. We estimate the first and second moments of $|V_j^{II}|$.

Fix $y \in A'(j)$. Following the arrows from y in **Stacks II** we perform a random walk until either we exit $A''(j)$, or we hit $A''(j) \cap \beta_j$. Therefore,

$$\begin{aligned} \mathbb{P}(y \in V_j^{II} \mid L, j \in J) &= \mathbb{P}(T_{A''(j) \cap \beta_j}[S^y] < \tau_{A''(j)}[S^y] \mid L, j \in J) \\ &\geq \mathbb{P}(T_{A(j) \cap \beta_j}[S^y] < \tau_{A''(j)}[S^y] \mid L, j \in J). \end{aligned} \tag{5.6}$$

The last expression is

$$\geq c \mathbb{P}(T_{A(j) \cap \beta_j}[S^y] < \infty \mid L, j \in J). \tag{5.7}$$

(One way to see this is by an argument similar to that of Lemma 3.6, where we let R count the number of crossings by the walk from a box $A^{**} \subset A''(j)$ to $\partial A''(j)$ before hitting $\beta_j \cap A(j)$, where each face of ∂A^{**} is at distance $m/16$ away from the corresponding face of $\partial A''(j)$.)

The Harnack inequality and Proposition 5.1 now implies, after summing over y in (5.6)–(5.7), that

$$\mathbb{E}(|V_j^{II}| \mid L, j \in J) \geq cc_1 m^d m^{4-d} = cm^4.$$

We now bound the second moment of $|V_j^{II}|$. If $x, y \in V_j^{II}$ occurs, then there exists a unique $w \in A''(j)$ with the property that cycle popping reveals three edge-disjoint paths: one from w to $A''(j) \cap \beta_j$, a second from x to w and a third from y to w . (We allow to have $x = w$ or $y = w$ or both.) When this event happens with a fixed w , we can reveal the paths by first following the arrows starting from w until $A''(j) \cap \beta_j$ is hit, then following

the arrows starting from x until w is hit, then following the arrows starting from y until w is hit. This shows that

$$\begin{aligned} & \mathbb{P}(x, y \in V_j^{II} \mid L, j \in J) \\ & \leq \sum_{w \in A''(j)} \mathbb{P}(T_{A''(j) \cap \beta_j}[S^w] < \infty \mid L, j \in J) \mathbb{P}(T_w[S^x] < \infty) \mathbb{P}(T_w[S^y] < \infty). \end{aligned} \quad (5.8)$$

Let $\tilde{A}(j) = Q(Y_{j,1}, (3m/2))$, and note that $\partial\tilde{A}(j)$ has distance at least cm from $A''(j) \cap \beta_j$, and also distance at least cm from $A'(j)$. We estimate separately the cases:

(a) $w \in A''(j) \setminus \tilde{A}(j)$; and

(b) $w \in A''(j) \cap \tilde{A}(j)$.

The sum of the terms in the right hand side of (5.8) corresponding to case (a) is at most:

$$\begin{aligned} & Cm^{2-d} \text{Cap}(A''(j) \cap \beta_j) \sum_{w \in A''(j) \setminus \tilde{A}(j)} \sum_{x, y \in A'(j)} G(x, w) G(y, w) \\ & \leq Cm^{2-d} m^2 m^2 m^d = Cm^8. \end{aligned}$$

The sum for case (b) is at most:

$$\begin{aligned} & Cm^{2-d} m^{2-d} m^d m^d \sum_{w \in A''(j) \cap \tilde{A}(j)} \mathbb{P}(T_{A''(j) \cap \beta_j}[S^w] < \infty) \\ & \leq Cm^4 \sum_{w \in \tilde{A}(j)} \mathbb{P}(T_{A''(j) \cap \beta_j}[S^w] < \tau_{\tilde{A}(j)}) \\ & \leq Cm^4 m^2 \text{Cap}(A''(j) \cap \beta_j) \\ & \leq Cm^8. \end{aligned}$$

Here the last line follows from $j \in J$ and Proposition 5.1.

The moment estimates for $|V_j^{II}|$ and the one-sided Chebyshev inequality yield:

$$\mathbb{P}(V_j^{II} \geq cm^4 \mid L, j \in J) \geq c > 0.$$

This completes the proof of the Lemma.

We can now complete the proof of Theorem 5.4. Choose $k \asymp \lambda^{-1/3}$ so that $\lambda N^4 \asymp km^4$. Then using Corollary 5.2, the conditional independence of $(V_j^{II})_{j \in J}$, and Lemma 5.5, for a suitably small $c_4 > 0$ we have

$$\begin{aligned} \mathbb{P}(|Q_N \cap \mathcal{U}_0| \leq \lambda N^4) & \leq C \exp(-ck) + \mathbb{E} \left(\mathbb{P} \left(V_j^{II} \geq c_3 m^4 \text{ for less than } c_4 k \text{ indices } j \in J \mid L \right) I[G(c_1, c_2, C_2)] \right) \\ & \leq C \exp(-c\lambda^{-1/3}). \end{aligned}$$

This completes the proof of the Theorem.

Theorem 5.7. *Assume $d \geq 5$ and let $\mathcal{U} = \mathcal{U}_{\mathbb{Z}^d}$. There exist $c > 0$ and C such that for all $\lambda > 0$ we have*

$$\mathbb{P}(|B_{\mathcal{U}}(0, n)| \leq \lambda n^2) \leq C \exp(-c\lambda^{-1/5}).$$

For the proof of this theorem, we assume the setting of Proposition 3.11, with $D = \mathbb{Z}^d$. Recall that $M_N = |\mathcal{E}_{\partial_i Q_N}^F L|$.

Lemma 5.8. *We have*

$$\mathbb{E}(M_N^k) \leq C_2^k k! N^{2k}.$$

Consequently, there exist $c > 0$ and C such that for all $\lambda > 0$ we have

$$\mathbb{P}(M_N \geq \lambda N^2) \leq C \exp(-c\lambda). \quad (5.9)$$

Remark 5.9. *If M_N^S is the length of a simple random walk path run until its first exit from Q_N then it is well known that M_N^S/N^2 has an exponential tail. However we do not have $M_N \leq M_N^S$, so need an alternative argument to obtain the bound (5.9).*

Proof. [Proof of Lemma 5.8.] We have

$$\begin{aligned} \mathbb{E}(M_N^k) &\leq \mathbb{E}(|S[0, \infty) \cap Q_N|^k) \\ &= k! \sum_{x_1, \dots, x_k \in Q_N} G(0, x_1) G(x_1, x_2) \dots G(x_{k-1}, x_k) \\ &\leq k! \left(\sum_{z \in Q_{2N}} G(0, z) \right)^k \\ &= C_2^k k! N^{2k}. \end{aligned}$$

To see the second statement:

$$\mathbb{P}(M_N \geq \lambda N^2) \leq \exp(-\lambda t N^2) \mathbb{E}(e^{t M_N}) \leq \exp(-\lambda t N^2) \frac{1}{1 - C_2 t N^2}.$$

Choosing $t = 1/(2C_2 N^2)$ completes the proof of the Lemma.

Proof. [Proof of Theorem 5.7] It is sufficient to prove the statement for $0 < \lambda < \lambda_0$ for some fixed λ_0 . Let us choose $N = \lambda^\alpha \sqrt{n}$ with some exponent $\alpha > 0$, that we will optimize over at the end of the proof. We have

$$\mathbb{P}(M_N \geq n/2) \leq C \exp\left(-c \frac{n}{2N^2}\right) = C \exp(-c\lambda^{-2\alpha}).$$

Condition on L , as in the proof of Theorem 5.4, and assume the event

$$\tilde{G} = G(c_1, c_2, C_1) \cap \{M_N < n/2\}.$$

We set

$$\lambda n^2 = c_3 k m^4 \asymp N m^3,$$

which means we pick m to be

$$m \asymp \sqrt{n} \lambda^{(1-\alpha)/3}.$$

Hence $N/m \asymp k \asymp \lambda^{(4\alpha-1)/3}$. Note that this implies that

$$\mathbb{P}(G(c_1, c_2, C_1)^c) \leq C \exp(-c(N/m)) = C \exp(-c\lambda^{(4\alpha-1)/3}).$$

Since we want $N/m \gg 1$, we impose the condition $0 < \alpha < 1/4$ on α .

For each $j \in J$, let

$$\begin{aligned}\tilde{V}_j^I &= \left\{ y \in A'(j) : \begin{array}{l} \text{cycle popping using \texttt{Stacks I} reveals a path} \\ \text{from } y \text{ to } L \text{ of length } \leq n/2 \end{array} \right\} \\ \tilde{V}_j^{II} &= \left\{ y \in A'(j) : \begin{array}{l} \text{cycle popping using \texttt{Stacks II} reveals a path} \\ \text{from } y \text{ to } A''(j) \cap \beta_j \text{ of length } \leq n/2 \end{array} \right\}\end{aligned}$$

Notice that $(\tilde{V}_j^I, \tilde{V}_j^{II})_{j \in J}$ are again conditionally independent, given L . The same proof as in Lemma 5.5 shows that we have $\tilde{V}_j^I \supset \tilde{V}_j^{II}$ for all $j \in J$.

In estimating $\mathbb{E}(\tilde{V}^{II})$ from below, we write

$$\begin{aligned}\mathbb{P}(y \in \tilde{V}_j^{II} \mid L, j \in J) &\geq \mathbb{P}(T_{A''(j) \cap \beta_j}[S^y] < \tau_{A''(j)}[S^y] \mid L, j \in J) \\ &\quad - \mathbb{P}(|\mathcal{E}_{\partial A''(j)}^F(S^y)| > n/2, T_{A''(j) \cap \beta_j}[S^y] \circ \Theta_{n/2} < \infty).\end{aligned}\tag{5.10}$$

The first term on the right hand side is $\geq cm^{4-d}$ due to (5.7) and $j \in J$. We now show that the subtracted term is $\leq C \exp(-cn/m^2)m^{4-d}$.

Note that we may restrict to $n/2 > 2m^2$ for convenience (although not needed for the claim), since our choice of m implies that $n \asymp m^2 \lambda^{-2(1-\alpha)/3}$, and we are considering small λ . Using the Markov property at time $n/2 - m^2$, the second term in the right hand side of (5.10) is at most

$$\mathbb{P}^y(\tau_{A''(j)} > n/2 - m^2) \sum_{z \in A''(j)} \mathbb{P}^z(T_{A''(j) \cap \beta_j} < \infty) \mathbb{P}^y(S(n/2) = z \mid \tau_{A''(j)} > n/2 - m^2).$$

The first probability can be bounded by $C \exp(-cn/m^2)$, by considering stretches of the walk of length m^2 , in each of which there is probability $\geq c > 0$ of exit from $A''(j)$. The conditional distribution of z is bounded above by cm^{-d} , due to the local CLT applied to $S(n/2 - m^2), \dots, S(n/2)$. Hence we are left to show that

$$\sum_{z \in A''(j)} \mathbb{P}^z(T_{A''(j) \cap \beta_j} < \infty) \leq m^4.$$

Let us write $\tilde{\beta}_j = A''(j) \cap \beta_j$, and $h(z) = \mathbb{P}^z(T_{\tilde{\beta}_j} < \infty)$. By a last exit decomposition $h(z) = \sum_{u \in \tilde{\beta}_j} G(z, u) e_{\tilde{\beta}_j}(u)$, where $e_{\tilde{\beta}_j}(u) = \mathbb{P}^u(T_{\tilde{\beta}_j}^+ = \infty)$. Therefore, we have

$$\begin{aligned}\sum_{z \in A''(j)} h(z) &= |\tilde{\beta}_j| + \sum_{z \in A''(j) \setminus \tilde{\beta}_j} h(z) \leq Cm^2 + \sum_{u \in \tilde{\beta}_j} \sum_{z \in A''(j)} G(z, u) e_{\tilde{\beta}_j}(u) \\ &\leq Cm^2 + Cm^2 \sum_{u \in \tilde{\beta}_j} e_{\tilde{\beta}_j}(u) = Cm^2 + Cm^2 \text{Cap}(\tilde{\beta}_j) \leq Cm^4,\end{aligned}$$

using that $|\tilde{\beta}_j|, \text{Cap}(\tilde{\beta}_j) \leq Cm^2$ when $j \in J$.

Hence we obtain that there exists $\lambda_0 = \lambda_0(d) > 0$, such that when $0 < \lambda \leq \lambda_0$, the right hand side of (5.10) is at least

$$cm^{4-d} - C \exp(-cn/m^2) m^{4-d} \geq cm^{4-d} - C \exp(-c\lambda^{-2(1-\alpha)/3}) m^{4-d} \geq cm^{4-d}.$$

It follows that $\mathbb{E}(|\tilde{V}_j^{II}| \mid L, j \in J) \geq cm^4$.

For the second moment, we simply estimate

$$\mathbb{E}((\tilde{V}_j^{II})^2 \mid L, j \in J) \leq \mathbb{E}((V_j^{II})^2 \mid L, j \in J) \leq Cm^8.$$

The one-sided Chebyshev inequality yields that for some $c_4 = c_4(d) > 0$ we have

$$\mathbb{P}(\tilde{V}_j^{II} \geq c_4 m^4 \mid L, j \in J) \geq c > 0.$$

This allows us to complete the proof as follows.

$$\begin{aligned} & \mathbb{P}(|B_{\mathcal{U}}(0, n)| \leq \lambda n^2) \\ & \leq \mathbb{P}(\tilde{G}^c) + \mathbb{P}\left(\tilde{G}, \sum_{j \in J} \tilde{V}_j^I \leq \lambda n^2\right) \\ & \leq \mathbb{P}(M_N > n/2) + \mathbb{P}(G(c_1, c_2, C_1)^c) + \mathbb{E}\left(\mathbb{P}\left(\sum_{j \in J} \tilde{V}_j^{II} < c_3 k m^4 \mid L\right); \tilde{G}\right) \\ & \leq C \exp(-c\lambda^{-2\alpha}) + C \exp(-c\lambda^{(4\alpha-1)/3}) + \exp(-c\lambda^{(4\alpha-1)/3}). \end{aligned}$$

We choose α , so that $-2\alpha = (4\alpha - 1)/3$, so $\alpha = 1/10$. This completes the proof of the Theorem.

Remark 5.10. *We note the following minor extension of Theorem 5.4, that is needed in [BHJ]. Similarly to Remark 5.3, since the arguments of Theorem 5.4 only rely on properties of $\mathcal{E}_{Q_N^c}^F L$, the result extends to the case when the component of the origin is connected to a fixed vertex $w \in \partial D$.*

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